I. A Short introduction to Stochastic Processes:

1. Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$

(i) Ω : a set = Sample space = set of all outcomes

(ii) $\mathcal{F}$: a family of subsets of Ω satisfying :

(1) $\Omega \in \mathcal{F}$, $\phi \in \mathcal{F}$

(2) If $A_n \in \mathcal{F}$, $n = 1, 2, \cdots$, Then $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$

(3) If $A \in \mathcal{F}$, Then $A^c \in \mathcal{F}$.

Then we call $\mathcal{F}$ a σ- algebra on Ω

The pair $(\Omega, \mathcal{F})$ called Measurable Space

(iii) $\mathbb{P}$: A mapping $\mathbb{P} : \mathcal{F} \xrightarrow{\text{into}} \mathbb{R}$ with properties :

(4) $0 \leq \mathbb{P}(A) \leq 1$, $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\phi) = 0$

(5) If $A_n \in \mathcal{F}$, $n = 1, 2, \cdots$ and $A_n \cap A_m = \phi$ for $n \neq m$

Then $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$. (Additive property)

is called a Probability Measure .

{Any set $A \in \mathcal{F}$ called a random event and $\mathbb{P}(A)$ called the probability of the event A .}

2. Random Variables

Roughly speaking, a r.v. is a function of outcome of an experiment with random outcome.

ex. Tossing a coin : $\Omega = \{H, T\}$, $X(H) = 10$, $X(T) = -10$

For a Mathematical Definition :

Recall the concept of a <u>Measurable function</u> :

Let $(\Omega, \mathcal{A})$ be a measurable space. Let $X(\omega), \omega \in \Omega$ be a function on Ω. The fn. $X(\omega)$ is called $\mathcal{A}$-measurable if $\forall\, B \in \mathcal{B}$ (Borel sets of $\mathbb{R}$), $X^{-1}(B) \in \mathcal{A}$ where

$$X^{-1}(B) = \{\omega : X(\omega) \in B\}.$$

<u>Definition of a r.v.</u> : Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability Space. A fn. $X : \Omega \to \mathbb{R}$ is called a (Real) random variable if it is $\mathcal{A}$-measurable, i.e $\forall\, B \in \mathcal{B}$ we have $X^{-1}(B) \in \mathcal{A}$.

$$\mathcal{A}_X := \{A : A = X^{-1}(B) \text{ for } B \in \mathcal{B}\} = \sigma(X)$$

and $\mathcal{A}_X \subset \mathcal{A}$.

<u>Distribution of Random Variables</u> :

<u>Definition</u> : The Distribution of a r.v. is the function $\mathbb{P}_X : \mathcal{B} \to \mathbb{R}^+$ defined by

$$\mathbb{P}_X(E) := \mathbb{P}(X \in E), \quad \forall\, E \in \mathcal{B}$$
$$= \mathbb{P}(\{\omega : X(\omega) \in E\}) = \mathbb{P}(X^{-1}(E))$$

Distribution function of a r.v.

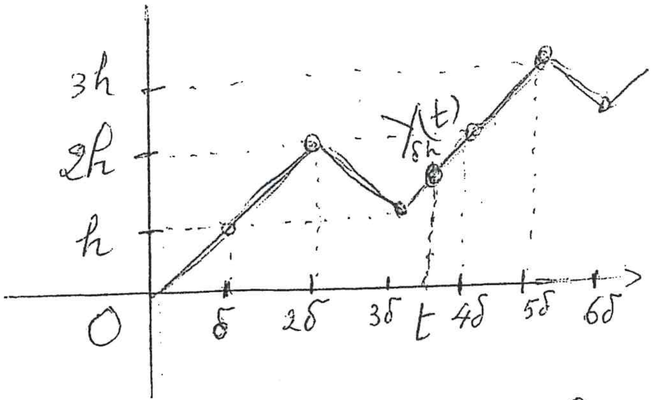
$$F_X(x) := \mathbb{P}_X\big((-\infty, x]\big) = \mathbb{P}(X \leq x) =$$
$$= \mathbb{P}\big(\{\omega : X(\omega) \leq x\}\big) = \mathbb{P}\big(X^{-1}(-\infty, x]\big)$$

Stochastic Processes :

A stochastic process X is a collection of random variables $X(t)$ indexed by a parameter $t \in T$ often taken to be time.

- Brownian Motion : We start with construction of ~~a~~ continuous stochastic process, the Brownian Motion as a random walk :

Brownian Motion as a limit of simple Random Walk:



$n\delta < t < (n+1)\delta$

$Y_{\delta,h}(t) = ?$

Write the eq. of the line throug the points:

$(n\delta, Y_{\delta,h}(n\delta))$ and $((n+1)\delta, Y_{\delta,h}((n+1)\delta))$

$$Y_{\delta,h}(t) = Y_{\delta,h}(n\delta) + \underbrace{\frac{Y_{\delta,h}((n+1)\delta) - Y_{\delta,h}(n\delta)}{(n+1)\delta - n\delta}}_{\text{slope of the line}}(t - n\delta)$$

$$= Y_{\delta,h}(n\delta) - Y_{\delta,h}(n\delta)\frac{t-n\delta}{\delta} + Y_{\delta,h}((n+1)\delta)\frac{t-n\delta}{\delta}$$

$$= Y_{\delta,h}(n\delta)\left[1 - \frac{t-n\delta}{\delta}\right] + Y_{\delta,h}((n+1)\delta)\frac{t-n\delta}{\delta}$$

$$Y_{\delta,h}(t) = \frac{(n+1)\delta - t}{\delta}Y_{\delta,h}(n\delta) + \frac{t-n\delta}{\delta}Y_{\delta,h}((n+1)\delta)$$

$$Y_{\delta,h}(n\delta) = X_1 + X_2 + \cdots + X_n$$

What is the limit of $Y_{\delta,h}(t)$ as $\delta, h \to 0$?

Compute the limit of the ch. fn. of $Y_{\delta,h}(t)$:

Let $t = n\delta$, $n = \dfrac{t}{\delta}$

$$\varphi_{Y_{\delta,h}}(\lambda) = \mathbb{E}\, e^{i\lambda Y_{\delta,h}(t)} = \mathbb{E}\, e^{i\lambda[X_1+X_2+\cdots+X_n]} =$$

$$= \left[\mathbb{E}\, e^{i\lambda X_1}\right]^n = \left[e^{i\lambda h}\frac{1}{2} + e^{-i\lambda h}\frac{1}{2}\right]^n =$$

$$= (\cos\lambda h)^n = (\cos\lambda h)^{t/\delta} \xrightarrow[\substack{\text{as } \delta\to 0 \\ h\to 0}]{} 1^\infty \ \nexists$$

Let us impose a certain relation between δ and h:

Let $u = (\cos\lambda h)^{1/\delta}$, $\ln u = \frac{1}{\delta}\ln(\cos\lambda h)$

$$\cos\lambda h \approx 1 - \frac{1}{2}\lambda^2 h^2 \quad , \quad \ln(1+x) \approx x$$

$$\ln(\cos(\lambda h)) \approx \ln\left(1 - \frac{1}{2}\lambda^2 h^2\right) \approx -\frac{1}{2}\lambda^2 h^2$$

$\therefore$ $\ln u \approx -\frac{1}{2\delta}\lambda^2 h^2$. So, $u \approx e^{-\frac{1}{2\delta}\lambda^2 h^2}$

$$\mathbb{E}\, e^{i\lambda Y_{\delta,h}(t)} \simeq e^{-\frac{1}{2\delta}t\lambda^2 h^2}.$$ Let $h^2 = \delta$, then:

$$\lim_{\delta\to 0} \mathbb{E}\, e^{i\lambda Y_{\delta,h}(t)} \cong e^{-\frac{1}{2}t\lambda^2}, \quad \lambda\in\mathbb{R}.$$

<u>Theorem</u> Let $Y_{\delta,h}(t)$ be the random walk, starting at 0 with jumps h and $-h$ equally likely at times $\delta, 2\delta, \ldots$ Assume that $h^2 = \delta$. Then for each $t \geq 0$, the limit:

$$B(t) = \lim_{\delta\to 0} Y_{\delta,h}(t)$$ exists in distribution. Moreover, we have:

$$\mathbb{E}\, e^{i\lambda B(t)} = e^{-\frac{1}{2}\lambda^2 t}, \quad \lambda\in\mathbb{R}.$$

Self-similar processes:

A stochastic process X_t, $t \geq 0$ is H-self-similar for some $H > 0$ if its fidis satisfy the condition:

$$\left(T^H B_{t_1}, \cdots, T^H B_{t_n}\right) \stackrel{d}{=} \left(B_{Tt_1}, \cdots, B_{Tt_n}\right)$$

for every $T > 0$, any choice of $t_i \geq 0$, $i = 1, 2, \cdots, n$, $n \geq 1$

Brownian Motion is $\frac{1}{2}$ self-similar, i.e.

$$\left(T^{1/2} B_{t_1}, \cdots, T^{1/2} B_{t_n}\right) \stackrel{d}{=} \left(B_{Tt_1}, \cdots, B_{Tt_n}\right)$$

$T > 0$, any choice of $t_i \geq 0$, $i = 1, 2, \cdots, n$ and $n \geq 1$

Hence its sample paths are <u>Nowhere differentiable</u>.

Suppose X_t is H-self-similar with stationary increments for some $H \in (0,1)$. Then for every fixed t_0,

$$\limsup_{t \downarrow t_0} \frac{X_t - X_{t_0}}{t - t_0} = \infty$$

Brownian Motion is of Unbounded Variation:

$$\sup_{\Pi} \sum \left| B_{t_i}(\omega) - B_{t_i - 1}(\omega) \right| = \infty .$$

- $B(0) = 0$: Standard Brownian Motion.
- $B(t)$ is a continuous process and has independent and stationary increments.

Other examples of Stochastic processes are :

<u>Levy Processes</u> : A stochastic process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is a <u>Levy process</u> if :

(1) $X(0) = 0$ a.s.

(2) For any $n \geq 1$ and $0 \leq t_0 < t_1 < \dots < t_n$, the rvs $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent. (independent increment property).

(3) The distribution of $X_{s+t} - X_s$ does not depend on s. Temporal <u>homogenety</u> OR stationary <u>increments</u> property.

(4) It is stochastically continuous : $\forall \varepsilon > 0$,

$$\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$$

(5) $X_t(\omega)$ is a.s. càdlàg OR RCLL.

<u>Additive Processes</u> Additive Process is a Stochastic Process satisfying the conditions (1), (2), (4) and (5)

Examples of Lévy Processes:

1. Brownian Motion, Poisson Process, - - -

Reproducing Kernel Hilbert Spaces (RKHS) Associated with Stochastic Processes:

Application of RKHS to stochastic processes concerning detection, filtering, parameter estimation and so on. Here we start with a Detection in Finite Dimensions:

1. Detection in Finite Dimensions

Let $m \in \mathbb{R}^n$ be a vector called the message. Let $w \in \mathbb{R}^n$ be a Gaussian withe noise i.e $w \in N(0, I)$. The detection problem is deciding whether the received signal $y \in \mathbb{R}^n$ represents only noise $y = w$ OR $y = m + w$.

Likelihood ratio test used.:

likelihood of y being just Noise is proportional to $\exp\{-\frac{1}{2}\|y\|^2\}$, while likelihood of $y = m + w$ is proportional to $\exp\{-\frac{1}{2}\|y - m\|^2\}$.

The message seemed to be present iff

$$\|y - m\| < \|y\|$$

The RKHS structure gives a link between the geometry $\langle \cdot, \cdot \rangle_K$ and the statistic of the Noise w.

2. The RKHS Associated with a Stochastic Process

We have seen that a Stochastic process X is a collection of random variables $X(t)$, $t \in T$
Here the inner product is defined by the formula: $\langle X(s), X(t) \rangle = \mathbb{E}(X(s) X(t))$

The coordinate system introduced by the RKHS approach assigns to each random variable U the function:

$$t \longrightarrow \mathbb{E}(U X(t)),$$

this coordinate system comes from taking the covariance function $\mathbb{E}(X(t) X(s)) = R(t,s)$, which is always positive semi-definite, to be the kernel of a RKHS $V \subset \mathbb{R}^T$
This construction arranges the individual points $X(t)$ in V according to the geometry

$$\langle X(s), X(t) \rangle = \mathbb{E}[X(s) X(t)].$$