Fundamental Properties of Reproducing Kernels and RKHSs

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Outline

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Fundamental Concepts of Hilbert space

- Vector Space, Inner Product and Inner Product Space, Metric
- Hilbert Space
- Review of Linear Operator
- Riesz Theorem
- Some important definitions and operators

- This part is mostly based on the book **"A Course in Functional Analysis**, Springer Verlag, Berlin - Heidelberg - New York, 1989" by J.B. Conway.

- **Vector Space** A vector space is a linear space that is closed under vector addition and scalar multiplication. More precisely, if we denote our linear space by $\mathcal{H}$ over the field $\mathbb{C}$, then it follows that
  - **(i)** if $x, y, z \in \mathcal{H}$, then

$$x + y = y + x \in \mathcal{H}, \quad x + (y + z) = (x + y) + z \in \mathcal{H};$$

  - **(ii)** if k is scalar, then $k\,x \in \mathcal{H}$.

- **Inner Product** Let $\mathcal{H}$ be a linear space over the complex field $\mathbb{C}$. An *inner product* on $\mathcal{H}$ is a two variable function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, satisfying
  1. $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for $f, g \in \mathcal{H}$.
  2. For $\alpha, \beta \in \mathbb{C}$ and $f, g, h \in \mathcal{H}$
     - $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$
     - $\langle f, \alpha g + \beta h \rangle = \overline{\alpha} \langle f, g \rangle + \overline{\beta} \langle f, h \rangle$
  3. $\langle f, f \rangle \geq 0$ for $f \in \mathcal{H}$ and $\langle f, f \rangle = 0 \iff f = 0$.

- **Pre-Hilbert Space** A vector space with an inner product is called a pre-Hillbert space (**Inner product space**) $\mathcal{H}$ over the complex field $\mathbb{C}$.

- **Norm** A norm on an inner product space $\mathcal{H}$ denoted by $\|\cdot\|$ is defined by

$$\|f\| = \langle f, f\rangle^{1/2} \quad \text{or} \quad \|f\|_{\mathcal{H}} = \langle f, f\rangle_{\mathcal{H}}^{1/2}$$

  where $f \in \mathcal{H}$ and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{H}}$ denotes the inner product on $\mathcal{H}$.

- **Properties of Norm** For all $f, g \in \mathcal{H}$, and $\lambda \in \mathbb{C}$, we have
  - $\|f\| \geq 0$. (The equality occurs only if $f = 0$).
  - $\|\lambda f\| = |\lambda|\|f\|$.

Schwartz Inequality For all $f, g \in \mathcal{H}$, it follows that

$$|\langle f, g \rangle| \leq \|f\| \|g\|. \tag{1}$$

In case if f and g are linearly dependent, then the inequality becomes equality.

Triangle Inequality For all $f, g \in \mathcal{H}$, it follows that

$$\|f + g\| \leq \|f\| + \|g\|. \tag{2}$$

In case if f and g are linearly dependent, then the inequality becomes equality.

Polarization Identity For all $f, g \in \mathcal{H}$, it follows that

$$\langle f, g \rangle = \frac{1}{4}(\|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - \|f - ig\|^2) \tag{3}$$

Parallelogram Identity For all $f, g \in \mathcal{H}$, it follows that

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2. \tag{4}$$

- **Metric** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the properties
  1. $d(x, y) \geq 0$ and $d(x, y) = 0$ only if $x = y$;
  2. $d(x, y) = d(y, x)$;
  3. $d(x, y) \leq d(x, z) + d(z, y)$;

  for all $x, y, z \in X$. Moreover the space (X, d) is the associated metric space.

- If we re-arrange the metric with its properties for the inner product space $\mathcal{H}$, then it follows that for all $f, g, h \in \mathcal{H}$ and for all $\lambda \in \mathbb{C}$, where d satisfies all requirements to be a metric, we have
  1. $d(f, g) \geq 0$ and equality occurs only if $f = g$.
  2. $d(f, g) = d(g, f)$.
  3. $d(f, g) \leq d(f, h) + d(h, g)$.
  4. $d(f - h, g - h) = d(f, g)$.
  5. $d(\lambda f, \lambda g) = |\lambda| \cdot d(f, g)$.

  So, every inner product space is a normed space, and hence also a metric space.

- **Linear Operator** A map L from a linear space to another linear space is called *linear operator* if

$$L(\alpha f + \beta g) = \alpha L f + \beta L g$$

is satisfied for all $\alpha,\ \beta \in \mathbb{C}$ and for all $f, g \in \mathcal{H}$. Some basic properties of the linear operators are given in the following.
  - **Continuous Operator** An operator L is said to be continuous if it is continuous at each point of its domain.
  - **Lipschitz Constant of a Linear Operator** If L is a linear operator from $\mathcal{H}$ to $\mathcal{G}$ where $\mathcal{H}$ and $\mathcal{G}$ are pre-Hilbert spaces, then the Lipschitz constant for L is its norm $\|L\|$ and it is defined by

$$\|L\| = \sup\{\|Lf\|_{\mathcal{G}}/\|f\|_{\mathcal{H}} : 0 \neq f \in \mathcal{H}\}. \tag{5}$$

Theorem

Let L be a linear operator from the pre-Hilbert spaces $\mathcal{H}$ to $\mathcal{G}$. Then the followings are mutually equivalent:

(i) *L is continuous*

(ii) *L is bounded, that is,*

$$\sup\{\|Lf\|_G : \|f\|_H \leq k\} \ < \infty$$

for $\quad 0 \leq k < \infty.$

(iii) *L is Lipschitz continuous, that is,*

$$\|Lf - Lg\|_G \leq \lambda \|f - g\|_{\mathcal{H}},$$

where $0 \leq \lambda < \infty$ and $f, g \in \mathcal{H}$.

Let $B(\mathcal{H}, \mathcal{G})$ be the collection of all continuous linear operators from the pre-Hilbert spaces $\mathcal{H}$ to $\mathcal{G}$.

- $B(\mathcal{H}, \mathcal{G})$ is a linear space with respect to the natural addition and scalar multiplication satisfying

$$(\alpha L + \beta M)f = \alpha Lf + \beta Mf,$$

  where L and M are linear operators, $f \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.

- Whenever $\mathcal{H} = \mathcal{G}$, then $B(\mathcal{H}, \mathcal{G})$ is denoted by $B(\mathcal{H})$.
- If $\mathcal{K}$ is another pre-Hilbert space, $L \in B(\mathcal{H}, \mathcal{G})$ and $K \in B(\mathcal{G}, \mathcal{K})$. Then the product

$$(KL)f = K(Lf) \quad \text{for} \quad f \in \mathcal{H} \in B(\mathcal{H}, \mathcal{K}).$$

  In addition,

  (i) $K(\xi L + \zeta M) = \xi KL + \zeta KM$
  (ii) $\|\xi L\| = |\xi| \cdot \|L\|$
  (iii) $\|L + M\| \leq \|L\| + \|M\|$ and
  (iv) $\|KL\| \leq \|K\|\|L\|$.

  are also satisfied ($\xi, \zeta \in \mathbb{C}$).

- **Normed Space** Let L, M are linear operators, $\mathcal{H}, \mathcal{G}$ pre-Hilbert spaces and $B(\mathcal{H}, \mathcal{G})$ is a metric space with respect to the translation invariant, positively homogenous distance function

  $$d(L, M) := \|L - M\|.$$

  Then $B(\mathcal{H}, \mathcal{G})$ is a normed space with the operator norm.
- **Normed Algebra** For each $K \in B(\mathcal{G}, \mathcal{K})$, the map $L \longmapsto KL$ becomes a continuous linear operator from $B(\mathcal{H}, \mathcal{G})$ to $B(\mathcal{H}, \mathcal{K})$. In particular, $B(\mathcal{H})$ is a normed algebra
- **Linear Form (or Linear Functional)** A linear operator from the pre-Hilbert space $\mathcal{H}$ to the scalar field $\mathbb{C}$ is called a *linear form* (or *linear functional*).

Definition

A pre-Hilbert space $\mathcal{H}$ is said to be a *Hilbert space* if it is complete in metric. In other words if f_n is a Cauchy sequence in $\mathcal{H}$, that is, if

$$\|f_n - f_m\| \longrightarrow 0 \quad \text{whenever} \quad n, m \to \infty,$$

then there is $f \in \mathcal{H}$ such that

$$\|f_n - f\| \longrightarrow 0 \quad \text{whenever} \quad n \to \infty.$$

Remark

- Every subspace of a pre-Hilbert space is also a pre-Hilbert space with respect to the induced inner product. However, for a subspace of a Hilbert space to be also a Hilbert space, it must be closed.
- Every finite dimensional subspace of a Hilbert space $\mathcal{H}$ is closed.

Hilbert Spaces

Theorem

Let (Ω, μ) denotes a measure space so that Ω is the union of subsets of finite positive measure and $L^2(\Omega, \mu)$ consists of all measurable functions $f(\omega)$ on Ω such that

$$\int_\Omega |f(\omega)|^2 \, d\mu(\omega) < \infty. \tag{6}$$

Then $L^2(\Omega, \mu)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle := \int_\Omega f(\omega) \, \overline{g(\omega)} \, d\mu(\omega). \tag{7}$$

Theorem (F. Riesz)

For each continuous linear functional φ on a Hilbert space $\mathcal{H}$, there exists uniquely $g \in \mathcal{H}$ such that

$$\varphi(f) = \langle f, g \rangle \text{ for } f \in \mathcal{H}. \tag{8}$$

- **Total Subset of a Hilbert Space** A subset $\mathcal{A}$ of a Hilbert space $\mathcal{H}$ is called *total* in $\mathcal{H}$ if 0 is the only element that is orthogonal to all elements of $\mathcal{A}$. In other words,

$$\mathcal{A}^{\perp} = \{0\}.$$

  As a result, $\mathcal{A}$ is total if and only if every element of $\mathcal{H}$ can be approximated by linear combinations of elements of $\mathcal{A}$.

- **Orthogonal Projection** If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, the map $f \mapsto f_{\mathcal{M}}$ gives a linear operator from $\mathcal{H}$ to $\mathcal{M}$ with norm ≤ 1. This operator is called as the *orthogonal projection* to $\mathcal{M}$ and denote it by $P_{\mathcal{M}}$.

- If I is the identity operator on $\mathcal{H}$, then $I - P_{\mathcal{M}}$ denotes the orthogonal projection to $\mathcal{M}^{\perp}$ and the relation

$$\|f\|^2 = \|P_{\mathcal{M}} f\|^2 + \|(I - P_{\mathcal{M}})f\|^2 \tag{9}$$

  is satisfied for all $f \in \mathcal{H}$.

Definition (Sesqui-linear Form)

A function $\Phi : \mathcal{H} \times \mathcal{G} \longrightarrow \mathbb{C}$ is a *sesqui-linear form* (or *sesqui-linear function*) if for $f, h \in \mathcal{H}$, $g, k \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{C}$,

$$\text{(i) } \Phi(\alpha f + \beta h, g) = \alpha \Phi(f, g) + \beta \Phi(h, g) \tag{10}$$

$$\text{(ii) } \Phi(f, \alpha g + \beta k) = \overline{\alpha} \Phi(f, g) + \overline{\beta} \Phi(f, k) \tag{11}$$

are satisfied where $\mathcal{H}$ and $\mathcal{G}$ are Hilbert spaces.

Remark: If $L \in B(\mathcal{H}, \mathcal{G})$, then the sesqui-linear form Φ is defined by

$$\Phi(f, g) = \langle Lf, g \rangle_G \tag{12}$$

is bounded in the sense that

$$|\Phi(f, g)| \leq \lambda \|f\|_{\mathcal{H}} \|g\|_{\mathcal{G}} \quad \text{for} \quad f \in \mathcal{H}, \; g \in \mathcal{G}, \tag{13}$$

where $\lambda \geq \|L\|$.

By the definitions of L and L^*, it follows that

$$\langle Lf, g\rangle_{\mathcal{G}} = \langle f, L^* g\rangle_{\mathcal{H}} \quad \text{for} \quad f \in \mathcal{H}, \; g \in \mathcal{G}. \tag{14}$$

- **Adjoint Operator** If $L \in B(\mathcal{H}, \mathcal{G})$, then the unique operator $L^* \in B(\mathcal{G}, \mathcal{H})$ satisfying

$$\Phi(f, g) = \langle f, L^* g\rangle_{\mathcal{H}} \;\; \text{for} \;\; f \in \mathcal{H}, g \in \mathcal{G} \tag{15}$$

  is called the *adjoint* of L.

- **Isometric Property** The adjoint operation is isometric if

$$\|L\| = \|L^*\| \quad \text{is satisfied.} \tag{16}$$

- **Remark** Let $\mathcal{H}, \mathcal{G}$ and $\mathcal{K}$ be Hilbert spaces and $K \in B(\mathcal{G}, \mathcal{K})$ and $L \in B(\mathcal{H}, \mathcal{G})$ be given. Then

$$KL \in B(\mathcal{H}, \mathcal{K}) \;\; \text{and} \;\; (KL)^* = L^* K^* \tag{17}$$

$$\text{Ker}(L) = (\text{Ran}(L^*))^\perp \text{ and } (\text{Ker}(L))^\perp = \text{Clos}\{\text{Ran}(L)^*\} \tag{18}$$

  where $\text{Ker}(L)$ is the kernel of L and $\text{Ran}(L)$ is the range of L.

- **Self-Adjoint Operator** A continuous linear operator L on a Hilbert space $\mathcal{H}$ is said to be *selfadjoint* if $L = L^*$.
- L is self adjoint if and only if the associated sesqui-linear form Φ is Hermitian.
- If L is a continuous selfadjoint operator, then

$$\|L\| = \sup\{|\langle Lf, f\rangle| : \|f\| \leq 1\}. \tag{19}$$

- **Positive Definite Operator** A self-adjoint operator $L \in B(\mathcal{H})$ is said to be *positive* (or *positive definite*) if

$$\langle Lf, f\rangle \geq 0 \quad \text{for all}\, f \in \mathcal{H}. \tag{20}$$

If $\langle Lf, f\rangle = 0$ only when $f = 0$, then L is said to be *strictly positive* (or, *strictly positive definite*).

- **Isometry** A linear operator U between Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$ is called *isometric* or an *isometry* if

$$\|Uf\|_{\mathcal{G}} = \|f\|_{\mathcal{H}} \quad \text{for} \quad f \in \mathcal{H} \tag{21}$$

  is satisfied, that is, it preserves the norm.

- For any positive operator $L \in B(\mathcal{H})$, the Schwartz inequality holds in the following sense

$$|\langle Lf, g\rangle|^2 \leq \langle Lf, f\rangle \cdot \langle Lg, g\rangle. \tag{22}$$

- The equation (21) implies that a continuous linear operator U is isometric if and only if $U^*U = I_{\mathcal{H}}$, in other words,

$$\langle Uf, Ug\rangle_{\mathcal{G}} = \langle f, g\rangle_{\mathcal{H}} \quad \text{for} \quad f, g \in \mathcal{H}, \tag{23}$$

  that is, U preserves inner product.

- **Unitary Operator** A surjective isometry linear operator $U : \mathcal{H} \longrightarrow \mathcal{H}$ is called a *unitary (operator)*.
- If $U \in B(H)$ is a unitary operator, then $U^* = U^{-1}$.
- **Partial Isometry** A continuous linear operator U between Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$ is called a *partial isometry* if

$$f \in (\text{Ker } U)^{\perp} = \mathrm{Ran}(U^*) \Rightarrow \|Uf\| = \|f\|.$$

  The space $(\text{Ker } U)^{\perp}$ and $Ran(U)$ are called the *initial space* of U and the *final space* of U, respectively.
- If U is partial isometry, then its adjoint U^* is also a partial isometry.
- **Theorem** Every continuous linear operator L on $\mathcal{H}$ admits a unique decomposition

$$L = U\tilde{L}, \tag{24}$$

  where $\tilde{L}$ is positive definite operator and U is a partial isometry with initial space the closure of $\mathrm{Ran}(\tilde{L})$.

Reproducing Kernels and RKHSs

- Definition and Basic Properties of Reproducing Kernel and RKHS
- Existence and Uniqueness of Reproducing Kernels and Associated RKHSs
- Properties and Some Important Theorems

This part is based on the following references:

- T. Ando, Reproducing Kernel Spaces and Quadratic Inequalities, Lecture Notes, Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo, Japan, 1987
- N. Aronszajn, Theory of reproducing kernels, TAMS Vol. 68, No.3, 1950, pp. 337-404
- S. Saitoh, Y. Sawano, Theory of Reproducing Kernels and Applications Springer, 2016.

Definition (Reproducing Kernel)

Let $\mathcal{H}$ be a Hilbert space of functions on a nonempty set X with the inner product $\langle f, g \rangle$ and norm

$$\|f\| = \langle f, f \rangle^{1/2}$$

for f and $g \in \mathcal{H}$. Then the complex valued function $K(x, y)$ of x and y in X is called a **reproducing kernel of $\mathcal{H}$** if

(i) for every $x \in X$, it follows that

$$K_x(\cdot) = K(x, \cdot) \in \mathcal{H}, \tag{25}$$

(ii) (reproducing property) for every $x \in X$ and every $f \in \mathcal{H}$,

$$f(x) = \langle f, K_x \rangle \tag{26}$$

RKHS

Notation

Let K be a reproducing kernel. Applying

$$f(x) = \langle f, K_x \rangle$$

to the function K_x at y, we get

$$K_x(y) = \langle K_x, K_y \rangle = K(x, y), \text{ for } x, y \in X. \tag{27}$$

Then, for any $x \in X$ we obtain

$$\|K_x\| = \langle K_x, K_x \rangle^{1/2} = K(x, x)^{1/2}. \tag{28}$$

Note: Observe that the subset $\{K_x\}_{x \in X}$ is **total** in $\mathcal{H}$, that is, its closed linear span coincides with $\mathcal{H}$. This follows from the fact that, if $f \in \mathcal{H}$ and $f \perp K_x$ for all $x \in X$, then

$$f(x) = \langle f, K_x \rangle = 0 \text{ for all } x \in X,$$

and hence f is the 0 element in $\mathcal{H}$. As a result, $\{0\}^{\perp} = \mathcal{H}$.

Definition (RKHS)

A Hilbert space $\mathcal{H}$ of functions on a set X is called a *reproducing kernel Hilbert space* (RKHS) if there exists a reproducing kernel K of $\mathcal{H}$.

Theorem (Existence of Reproducing Kernel)

There exists a reproducing kernel K for a Hilbert space $\mathcal{H}$ of functions on X, if and only if for all $x \in X$, the linear functional

$$\mathcal{H} \ni f \longmapsto f(x)$$

of evaluation at x, is bounded on $\mathcal{H}$.

$$i.e. |\langle f, K_x \rangle| = |f(x)| \leq C \|f(x)\|_{\mathcal{H}} \; \forall f \in \mathcal{H}$$

Proof: Suppose that K is the reproducing kernel for $\mathcal{H}$. By the reproducing property and the Schwarz inequality of the scalar product, for all $x \in X$,

$$\begin{aligned} |f(x)| = |\langle f, K_x \rangle| \leq \|f(x)\| \|K_x\| &= \|f(x)\| \langle K_x, K_x \rangle^{1/2} \\ &= \|f(x)\| K(x,x)^{1/2} \\ &= C\|f(x)\| \end{aligned}$$

$\forall f \in \mathcal{H}$ with $C = K(x,x)^{1/2}$.
Conversely, if for all $x \in X$ the evaluation $\mathcal{H} \ni f \mapsto f(x)$ is a bounded linear functional on $\mathcal{H}$, then by the Riesz Representation Theorem, for all $x \in X$ there exists a function g_x belonging to $\mathcal{H}$ such that

$$f(x) = \langle f, g_x \rangle.$$

If we put K_x instead of g_x, then for all $y \in X$, we get $K_x(y) = g_x(y)$. Hence K is a reproducing kernel for $\mathcal{H}$.

Theorem (Uniqueness of Reproducing Kernel)

If a Hilbert space $\mathcal{H}$ of functions on a set X admits a reproducing kernel K, then this reproducing kernel K is uniquely determined by the Hilbert space $\mathcal{H}$.

Proof: Let $\mathcal{H}$ be a RKHS with two reproducing kernels K and L. For any two points $x, y \in X$, we need to show that $K(x, y) = L(x, y)$. Using the properties of RKHS, $K_x, L_x \in \mathcal{H}$. Then

$$\begin{aligned}\|K_x - L_x\|_{\mathcal{H}}^2 &= \langle K_x - L_x, K_x - L_x\rangle_{\mathcal{H}} \\ &= \langle K_x - L_x, K_x\rangle_{\mathcal{H}} - \langle K_x - L_x, L_x\rangle_{\mathcal{H}} \\ &= (K_x - L_x)(x) - (K_x - L_x)(x) \\ &= 0\end{aligned}$$

Since $\mathcal{H}$ is a Hilbert space, only the zero function has a norm equal to 0. This shows that

$$K_x = L_x$$

as and hence

$$K_x(y) = L_x(y) \quad \forall y \in X \qquad \Longrightarrow \qquad K(x, y) = L(x, y).$$

Theorem (Uniqueness of RKHS)

For any positive definite kernel K on X, there exists a unique Hilbert space $\mathcal{H}_K$ of functions on X with reproducing kernel K.

By the above theorem, if $\mathcal{H}$ and $\mathcal{G}$ are two RKHS having the same reproducing kernel K, then they are equal, i.e. $\mathcal{H} = \mathcal{G}$.

(For the proof, see Ando [1], Aronszajn [2] or Saitoh, [7]).

Hermitian and Positive Definite Kernel

Let X be an arbitrary set and K be a kernel on X, that is,

$$K\colon X \times X \to \mathbb{C}.$$

The kernel K is **Hermitian** if for any finite set of points $\{y_1, \ldots, y_n\} \subseteq X$ we have

$$\sum_{i,j=1}^{n} \bar{\epsilon}_j \epsilon_i K(y_j, y_i) \in \mathbb{R}.$$

K is **positive definite**, if for any complex numbers $\epsilon_1, \ldots, \epsilon_n$, we have

$$\sum_{i,j=1}^{n} \bar{\epsilon}_j \epsilon_i K(y_j, y_i) \geq 0.$$

Note: The last inequality can be denoted by $[K(x, y)] \geq 0$ on X, or simply by $K \geq 0$ on X, or equivalently, we say that K is a positive definite matrix in the sense of **E. H. Moore**.

RKHS

Remark

From the previous inequality, it follows that for any finitely supported family of complex numbers $\{\epsilon_x\}_{x\in X}$ we have

$$\sum_{x,y\in X} \bar{\epsilon}_y \epsilon_x K(y,x) \geq 0. \tag{29}$$

Theorem

The reproducing kernel K of a RKHS $\mathcal{H}$ is a positive definite matrix (in the sense of E.H. Moore)

Note: In the sense of Moore, a positive definite matrix satisfies the following:

- It is conjugate symmetric, that is, $K(x,y) = \overline{K(y,x)}$, for all $x, y \in \mathcal{H}$
- $K(x,x) \geq 0 \quad$ for all $\quad x \in \mathcal{H}$
- $|K(x,y)|^2 \leq K(x,x)K(y,y) \quad$ for all $\quad x, y \in \mathcal{H}$

RKHS

Proof: For arbitrary finite set of points $\{y_1, \cdots, y_n\} \subseteq X$ and any complex numbers $\epsilon_1, \ldots, \epsilon_n$, we have

$$
\begin{aligned}
0 \leq \| \sum_{i=1}^{n} \epsilon_i K_{y_i} \|^2 &= \langle \sum_{i=1}^{n} \epsilon_i K_{y_i}, \sum_{j=1}^{n} \epsilon_j K_{y_j} \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_i \overline{\epsilon_j} \langle K_{y_i}, K_{y_j} \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_i \overline{\epsilon_j} K(y_i, y_j)
\end{aligned}
$$

Hence

$$
\sum_{i,j=1}^{n} \bar{\epsilon}_j \epsilon_i K(y_j, y_i) \geq 0.
$$

i.e. K is positive definite.

Properties of RKHS

Properties of RKHS

Given a reproducing kernel Hilbert space $\mathcal{H}$ and its kernel $K(y,x)$ on X, then for all $x, y \in X$ we have

(i) $K(y,y) \geq 0$.

(ii) $K(y,x) = \overline{K(x,y)}$.

(iii) $|K(y,x)|^2 \leq K(y,y)K(x,x)$, (Schwarz Inequality).

(iv) Let $x_0 \in X$. Then the following statements are equivalent:

   (a) $K(x_0, x_0) = 0$.
   (b) $K(y, x_0) = 0$ for all $y \in X$.
   (c) $f(x_0) = 0$ for all $f \in \mathcal{H}$.

Proof :
(i) and (ii) can be easily seen from the reproducing and norm properties (27) and (28), respectively.
For (iii) we use the Schwarz Inequality in $\mathcal{H}$. It follows that

$$\begin{aligned} |K(y,x)|^2 &= |\langle K_y, K_x\rangle|^2 \\ &\leq \|K_y\|\|K_x\|\|K_y\|\|K_x\| \\ &= \|K_y\|^2\|K_x\|^2 \\ &= \langle K_y, K_y\rangle\langle K_x, K_x\rangle \\ &= K(y,y)K(x,x) \end{aligned}$$

As for (iv), it follows by (iii) that $K(x_0, x_0) = 0$ is equivalent with $K(y, x_0) = 0$ for all $y \in X$. Further, by the reproducing property we have that $K(y, x_0) = 0$ for all $y \in X$ if and only if $f(x_0) = 0$, for all f.

Notation

The Hilbert space with reproducing kernel K is denoted by

$$\mathcal{H}_K(X).$$

Moreover, the norm is denoted by

$$\| \cdot \|_K = \| \cdot \|_{\mathcal{H}_K}$$

and the inner product is denoted by

$$\langle \cdot, \cdot \rangle_K = \langle \cdot, \cdot \rangle_{\mathcal{H}_K}.$$

RKHS

Theorem

Every sequence of functions $(f_n)_{n\geq 1}$ that converges strongly to a function f in $\mathcal{H}_K(X)$, converges also in the pointwise sense, i.e., for any point $x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

In addition, this convergence is uniform on every subset of X on which $x \mapsto K(x,x)$ is bounded.

Proof: For $x \in X$, by the reproducing property and the Schwarz Inequality,

$$\begin{aligned} |f(x) - f_n(x)| &= |\langle f, K_x\rangle - \langle f_n, K_x\rangle| \\ &= |\langle f - f_n, K_x\rangle| \\ &\leq \|f - f_n\| \cdot \|K_x\| \\ &= \|f - f_n\| K(x,x)^{1/2} \end{aligned}$$

Hence $\lim f_n(x) = f(x)$, for any point $x \in X$. Moreover, it is clear from the above inequality that this convergence is uniform on every subset of X on which $x \mapsto K(x,x)$ is bounded.

Operations with RKHSs

Theorem

Let $K^{(0)}$ be the restriction of the positive definite kernel K to a nonempty subset X_0 of X and let $\mathcal{H}_{K^{(0)}}(X)$ and $\mathcal{H}_K(X)$ be the RKHS corresponding to $K^{(0)}$ and K, respectively. Then

$$\mathcal{H}_{K^{(0)}}(X_0) = \{f|_{X_0} : f \in \mathcal{H}_K(X)\} \tag{30}$$

and

$$\|h\|_{K^{(0)}} = \min\{\|f\|_K : f|_{X_0} = h\} \quad \text{for all } h \in \mathcal{H}_{K^{(0)}}(X_0). \tag{31}$$

Remark

If $K^{(1)}(y,x)$ and $K^{(2)}(y,x)$ are two positive definite kernels, then

$$K(y,x) = K^{(1)}(y,x) + K^{(2)}(y,x)$$

is also a positive definite kernel.

Operations with RKHSs

Theorem

The tensor product Hilbert space

$$\mathcal{H}_{K^{(1)}}(X) \otimes \mathcal{H}_{K^{(2)}}(X)$$

is a RKHS on $X \times X$.

Take $g \in \mathcal{H}_{K^{(1)}}(X)$, $h \in \mathcal{H}_{K^{(2)}}(X)$ and $x, x' \in X$. It follows

$$(g \otimes h)(x, x') = g(x)h(x') = \langle g, K_x^{(1)} \rangle \langle h, K_{x'}^{(2)} \rangle = \langle g \otimes h, K_x^{(1)} \otimes K_{x'}^{(2)} \rangle$$

which shows that the tensor product Hilbert space $\mathcal{H}_{K^{(1)}}(X) \otimes \mathcal{H}_{K^{(2)}}(X)$ is a RKHS on $X \times X$.
Consider the map $\varphi : X \longrightarrow \mathcal{H}_{K^{(1)}}(X) \otimes \mathcal{H}_{K^{(2)}}(X)$ defined by $x \mapsto K_x^{(1)} \otimes K_x^{(2)}$. Then

$$\begin{aligned} K(y, x) = \langle \varphi_x, \varphi_y \rangle &= \langle K_x^{(1)} \otimes K_x^{(2)}, K_y^{(1)} \otimes K_y^{(2)} \rangle = \langle K_x^{(1)}, K_y^{(1)} \rangle \cdot \langle K_x^{(2)}, K_y^{(2)} \rangle \\ &= K^{(1)}(y, x) \cdot K^{(2)}(y, x) \quad \text{for} \quad x, y \in X. \end{aligned}$$

Hence the pointwise product of two positive definite kernels is again a positive definite kernel.

Bergman Space and Its Kernel

Definition (Bergman Space)

The space of all analytic functions f on Ω for which

$$\iint_{\Omega} |f(z)|^2 dxdy < \infty, \quad (z = x + iy)$$

is satisfied, is called the *Bergman space* on Ω and denoted by $A^2(\Omega)$.

Definition (Bergman Kernel)

$A^2(\Omega)$ is a *RKHS* with respect to the inner product

$$\langle f, g \rangle \equiv \langle f, g \rangle_{\Omega} := \iint_{\Omega} f(z)\overline{g(z)} dxdy$$

and its kernel is called the *Bergman kernel* on Ω and denoted by $B^{(\Omega)}(w, z)$.

Bergman Kernel

Bergman Kernel For the Unit Disc

The Bergman kernel for the open unit disc $\mathbb{D}$ is given by

$$B^{(\mathbb{D})}(w,z) = \frac{1}{\pi}\frac{1}{(1-w\overline{z})^2} \qquad \text{for} \quad w,z \in \mathbb{D}. \tag{32}$$

Bergman Kernel of a Simply Connected Domain

The Bergman kernel of a simply connected domain $\Omega(\neq \mathbb{C})$ is given by

$$B^{(\Omega)}(w,z) = \frac{1}{\pi}\frac{\varphi'(w)\overline{\varphi'(z)}}{\left(1-\varphi(w)\overline{\varphi(z)}\right)^2} \quad \text{for} \quad w,z \in \Omega, \tag{33}$$

where φ is any conformal mapping function from Ω onto $\mathbb{D}$.

References

- T. Ando, *Reproducing Kernel Spaces and Quadratic Inequalities*, Lecture Notes, Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo, Japan, 1987.
- N. Aronszajn, *Theory of reproducing kernels*, TAMS Vol. 68, No.3, 1950, pp. 337-404.
- J.B. Conway, *A Course in Functional Analysis,* Springer Verlag, Berlin - Heidelberg - New York, 1989.
- P.L. Duren, A. Schuster, *Bergman Spaces*, Amer. Math. Soc., Providence R.I. 2004.
- P. Koosis, *Introduction to H_p Spaces*, Cambridge Mathematical Press, Cambridge, 1970.
- B. Okutmustur,, *A Survey on Hilbert Spaces and Reproducing Kernels in Functional Calculus, London: Intech Open, 2020, pp 61-77*
- S. Saitoh, Y. Sawano, *Theory of Reproducing Kernels and Applications* Springer, 2016.